GCE Examinations

Further Pure Mathematics Module FP3

Advanced Subsidiary / Advanced Level

Paper A

Time: 1 hour 30 minutes

Instructions and Information

Candidates may use any calculator except those with a facility for symbolic algebra and / or calculus.

Full marks may be obtained for answers to ALL questions.

Mathematical and statistical formulae and tables are available.

This paper has 8 questions.

Advice to Candidates

You must show sufficient working to make your methods clear to an examiner. Answers without working will gain no credit.

Written by Rosemary Smith & Shaun Armstrong

1. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \; [\mathbf{r} - (-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})] \times (\mathbf{i} + \mathbf{k}) = 0,$$

$$l_2 : \; [\mathbf{r} - (\mathbf{i} + \mathbf{j} + 4\mathbf{k})] \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0.$$

(a) Find $(\mathbf{i} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. **(3 marks)**

(b) Find the shortest distance between l_1 and l_2. **(3 marks)**

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2. Prove by induction that, for all $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (r^2 + 1)r! = n(n + 1)!$$ **(6 marks)**

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3. *(a)* Solve the equation

$$z^3 + 27 = 0,$$

giving your answers in the form $re^{i\theta}$ where $r > 0$, $-\pi < \theta \leq \pi$. **(5 marks)**

(b) Show the points representing your solutions on an Argand diagram. **(2 marks)**

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4. $$\mathbf{A} = \begin{pmatrix} 2 & a \\ 2 & b \end{pmatrix}.$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$.

(a) Find the value of a and the value of b. **(4 marks)**

Using your values of a and b,

(b) for each eigenvalue, find a corresponding eigenvector, **(3 marks)**

(c) find a matrix $\mathbf{P}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. **(2 marks)**

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5. $$(1+x^2)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \quad \text{and} \quad y=1,\ \frac{dy}{dx}=1 \text{ at } x=-1.$$

Find a series solution of the differential equation in ascending powers of $(x + 1)$ up to and including the term in $(x + 1)^4$.

(11 marks)

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6. The variable y satisfies the differential equation

$$\frac{d^2y}{dx^2}=x\frac{dy}{dx}+y^2 \quad \text{with} \quad y=1.2 \text{ at } x=0.1 \quad \text{and} \quad y=0.9 \text{ at } x=0.2$$

Use the approximations $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1-y_{-1}}{2h}$ and $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1-2y_0+y_{-1}}{h^2}$ with a step length of 0.1 to estimate the values of y at $x = 0.3$ and $x = 0.4$ giving your answers to 3 significant figures.

(11 marks)

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7. $$\mathbf{M}=\begin{pmatrix} 2 & 1 & 1 \\ k & 4 & 3 \\ -1 & k & 2 \end{pmatrix}.$$

(a) Find the determinant of **M** in terms of k. **(2 marks)**

(b) Prove that **M** is non-singular for all real values of k. **(2 marks)**

(c) Given that $k = 3$, find $\mathbf{M}^{-1}$, showing each step of your working. **(4 marks)**

When $k = 3$ the image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by **M** is the vector $\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$.

(d) Find the values of a, b and c. **(3 marks)**

Turn over

8. A transformation T from the z-plane to the w-plane is defined by

$$w = \frac{z+1}{\mathrm{i}z-1}, \quad z \neq -\mathrm{i},$$

where $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$ and x, y, u and v are real.

T transforms the circle $|z| = 1$ in the z-plane onto a straight line L in the w-plane.

(a) Find an equation of L giving your answer in terms of u and v. **(5 marks)**

(b) Show that T transforms the line $\text{Im}\, z = 0$ in the z-plane onto a circle C in the w-plane, giving the centre and radius of this circle. **(6 marks)**

(c) On a single Argand diagram sketch L and C. **(3 marks)**

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