GCE Examinations

Advanced Subsidiary / Advanced Level

# Further Pure Mathematics
# Module FP3

Paper D

# MARKING GUIDE

This guide is intended to be as helpful as possible to teachers by providing concise solutions and indicating how marks should be awarded. There are obviously alternative methods that would also gain full marks.

Method marks (M) are awarded for knowing and using a method.

Accuracy marks (A) can only be awarded when a correct method has been used.

(B) marks are independent of method marks.

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# FP3 Paper D – Marking Guide

**1.** assume true for $n = k$ $\therefore$ $\frac{d^k y}{dx^k} = \frac{k!}{(1-x)^{k+1}}$

$\therefore$ $\frac{d^{k+1} y}{dx^{k+1}} = -(k+1)k!(-1)(1-x)^{-(k+2)}$ — M1 A1

$= \frac{k!(k+1)}{(1-x)^{k+2}} = \frac{(k+1)!}{(1-x)^{[k+1]+1}}$ — M1 A1

$\therefore$ true for $n = k + 1$ if true for $n = k$

if $n = 1$, $\frac{d^1 y}{dx^1} = \frac{1!}{(1-x)^{1+1}} = \frac{1}{(1-x)^2}$ — M1

$y = \frac{1}{1-x}$, $\frac{dy}{dx} = -(-1)(1-x)^{-2} = \frac{1}{(1-x)^2}$ $\therefore$ true for $n = 1$ — A1

$\therefore$ by induction true for $n \in \mathbb{Z}^+$ — A1 **(7)**

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**2.** *(a)* $\frac{y_1 - y_{-1}}{2h} = x_0^2 + y_0 + 2$ — M1

$y_1 = 2hx_0^2 + 2hy_0 + 4h + y_{-1}$ or $y_2 = 2hx_1^2 + 2hy_1 + 4h + y_0$ — A1

$x_0 = 0, x_1 = h, x_2 = 2h; y_0 = 0, y_1 = 2h, y_2 = ?$

$y_2 = 2h(h^2) + 2h(2h) + 4h + 0 = 2h^3 + 4h^2 + 4h$ — M1 A1

*(b)* $y_3 = 2hx_2^2 + 2hy_2 + 4h + y_1$ — B1

$x_1 = h, x_2 = 2h, x_3 = 3h;\ y_1 = 2h, y_2 = 2h^3 + 4h^2 + 4h, y_3 = ?$

$y_3 = 2h(2h)^2 + 2h(2h^3 + 4h^2 + 4h) + 4h + 2h$ — M1

$= 8h^3 + 4h^4 + 8h^3 + 8h^2 + 6h = 2h(2h^3 + 8h^2 + 4h + 3)$ — A1

*(c)* $h = 0.1, y_3 = 0.2(0.002 + 0.08 + 0.4 + 3) = 0.6964$ — M1 A1 **(9)**

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**3.** *(a)* using quad. formula $z^3 = \frac{\sqrt{3} \pm \sqrt{3-4}}{2}$ — M1

$\therefore z^3 = \frac{\sqrt{3}}{2} \pm \frac{1}{2}\text{i}$ — M1 A1

*(b)* if $z^3 = \frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$, $(re^{i\theta})^3 = 1e^{i\frac{\pi}{6}}$ — M1 A1

$r^3 = 1$ so $r = 1$

$3\theta = 2n\pi + \frac{\pi}{6}$ — M1

$n = -1, 0, 1$ gives $\theta = -\frac{11\pi}{18}, \frac{\pi}{18}, \frac{13\pi}{18}$ — A1

if $z^3 = \frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}$, $(re^{i\theta})^3 = 1e^{-i\frac{\pi}{6}}$ — M1

$r = 1, 3\theta = 2n\pi - \frac{\pi}{6}$

$n = -1, 0, 1$ gives $\theta = -\frac{13\pi}{18}, -\frac{\pi}{18}, \frac{11\pi}{18}$ — A1

$\therefore z = e^{\pm i\frac{\pi}{18}}, e^{\pm i\frac{11\pi}{18}}, e^{\pm i\frac{13\pi}{18}}$ — A1 **(10)**

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**4.** *(a)* $e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \ldots$ M1 A1

*(b)* $(1 + 2x)^{-1}$

$= 1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(2x)^3 + \frac{(-1)(-2)(-3)(-4)}{4\times3\times2}(2x)^4 + \ldots$ M1

$= 1 - 2x + 4x^2 - 8x^3 + 16x^4 + \ldots$ A1

$\frac{e^{x^2}}{1+2x} = (1 + x^2 + \frac{1}{2}x^4 + \ldots)(1 - 2x + 4x^2 - 8x^3 + 16x^4 + \ldots)$ M1

$= 1 - 2x + 4x^2 - 8x^3 + 16x^4 + x^2 - 2x^3 + 4x^4 + \frac{1}{2}x^4 + \ldots$ M1

$= 1 - 2x + 5x^2 - 10x^3 + \frac{41}{2}x^4 + \ldots$ A1

*(c)* area $\approx \int_0^{0.2} 1 - 2x + 5x^2 - 10x^3 + \frac{41}{2}x^4 \, dx$ M1

$= [\, x - x^2 + \frac{5}{3}x^3 - \frac{5}{2}x^4 + \frac{41}{10}x^5 \,]_0^{0.2}$ A1

$= \frac{1}{5} - \frac{1}{25} + \frac{1}{75} - \frac{1}{250} + \frac{41}{31250} = 0.171$ (3sf) M1 A1 **(11)**

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**5.** *(a)* det $\mathbf{A} = 2(2 + 1) - a(1 + 3) + 1(1 - 6) = 6 - 4a - 5 = 1 - 4a$ M1 A1

$\mathbf{A}$ is non-singular for $a \neq \frac{1}{4}$ A1

matrix of cofactors: $\begin{pmatrix} 3 & -4 & -5 \\ 1-a & -1 & 3a-2 \\ -a-2 & 3 & 4-a \end{pmatrix}$ M1 A1

$\therefore \mathbf{A}^{-1} = \frac{1}{1-4a}\begin{pmatrix} 3 & 1-a & -a-2 \\ -4 & -1 & 3 \\ -5 & 3a-2 & 4-a \end{pmatrix}$ M1 A1

*(b)* $a = -1, \ \mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 2 & -1 \\ -4 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$ B1

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 2 & -1 \\ -4 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ -10 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ M1 A1

$\therefore$ position vector of $P$ is $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ A1 **(11)**

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| **6.** | *(a)* | $\Pi_1 : \mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 5, \quad \Pi_2 : \mathbf{r}.(\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -2$ | | |
| | | $\therefore (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}).(\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = \sqrt{14}\sqrt{18}\cos\theta$ | M1 | |
| | | $2 - 4 + 3 = 1 = \sqrt{14}\sqrt{18}\cos\theta$ | M1 A1 | |
| | | $\therefore \cos\theta = \frac{1}{\sqrt{14}\sqrt{18}}$ giving $\theta = 86°$ (nearest degree) | A1 | |
| | *(b)* | $\Pi_1 : \mathbf{r}.\frac{2\mathbf{i}-\mathbf{j}+3\mathbf{k}}{\sqrt{14}} = \frac{5}{\sqrt{14}}$ | B1 | |
| | | plane parallel to $\Pi_1$ through $A$: | | |
| | | $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 4 - 1 - 6 = -3$ | M1 | |
| | | $\therefore \mathbf{r}.\frac{2\mathbf{i}-\mathbf{j}+3\mathbf{k}}{\sqrt{14}} = \frac{-3}{\sqrt{14}}$ | A1 | |
| | | $\therefore$ distance $A$ to $\Pi_1 = \frac{8}{\sqrt{14}}$ or $\frac{4}{7}\sqrt{14}$ | A1 | |
| | *(c)* | $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 4 & 1 \end{vmatrix}$ | M1 | |
| | | $= \mathbf{i}(-1 - 12) - \mathbf{j}(2 - 3) + \mathbf{k}(8 + 1) = -13\mathbf{i} + \mathbf{j} + 9\mathbf{k}$ | A1 | |
| | | $\Pi_3 : \mathbf{r}.(-13\mathbf{i} + \mathbf{j} + 9\mathbf{k}) = (4\mathbf{j} - \mathbf{k}).(-13\mathbf{i} + \mathbf{j} + 9\mathbf{k}) = 4 - 9 = -5$ | M1 | |
| | | $\therefore \Pi_3 : \mathbf{r}.(-13\mathbf{i} + \mathbf{j} + 9\mathbf{k}) = -5$ | A1 | |
| | | giving $-13x + y + 9z = -5$ or $13x - y - 9z = 5$ | A1 | **(13)** |

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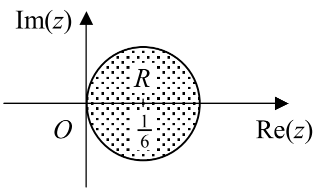
| **7.** | *(a)* | $\text{Re}(z) = 5 \quad \therefore u + \text{i}v = \frac{1}{5 - \text{i}y - 2} = \frac{1}{3 - \text{i}y}$ | M1 | |
| | | $(u + \text{i}v)(3 - \text{i}y) = 1$ | A1 | |
| | | $3u + vy + \text{i}(3v - uy) = 1$ | | |
| | | $\therefore 3u + vy = 1;\ 3v - uy = 0$ | M1 | |
| | | giving $y = \frac{1-3u}{v} = \frac{3v}{u}$ | A1 | |
| | | $\therefore u - 3u^2 = 3v^2;\quad u^2 + v^2 - \frac{1}{3}u = 0$ | M1 | |
| | | $(u - \frac{1}{6})^2 + v^2 = \frac{1}{36}$ | A1 | |
| | | $\therefore$ circle, centre $\frac{1}{6} + 0\text{i}$, radius $\frac{1}{6}$ | A1 | |
| | *(b)* | e.g. if $z = 6$, $z^* = 6$, $w = \frac{1}{4}$ which is inside circle | B1 | |
| | |  | B2 | |
| | *(c)* | $\arg(z - 2) = \frac{\pi}{4} \quad \therefore \arg(z^* - 2) = -\frac{\pi}{4}$ | M1 A1 | |
| | | $\therefore \arg w = \arg 1 - \arg(z^* - 2) = 0 - (-\frac{\pi}{4}) = \frac{\pi}{4}$ | M1 | |
| | | image is half-line $\arg w = \frac{\pi}{4}$ | A1 | **(14)** |

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Total **(75)**

# Performance Record – FP3 Paper D

| Question no. | **1** | **2** | **3** | **4** | **5** | **6** | **7** | Total |
|---|---|---|---|---|---|---|---|---|
| Topic(s) | proof by induction | step-by-step soln. of diff. eqn. | complex nos. | Maclaurin series | matrices, inverse | vectors | complex trans. | |
| Marks | 7 | 9 | 10 | 11 | 11 | 13 | 14 | 75 |
| Student | | | | | | | | |
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