GCE Examinations

Advanced Subsidiary / Advanced Level

Further Pure Mathematics
Module FP3

Paper C

MARKING GUIDE

This guide is intended to be as helpful as possible to teachers by providing concise solutions and indicating how marks should be awarded. There are obviously alternative methods that would also gain full marks.

Method marks (M) are awarded for knowing and using a method.

Accuracy marks (A) can only be awarded when a correct method has been used.

(B) marks are independent of method marks.

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FP3 Paper C – Marking Guide

1.	*(a)*	$\frac{y_1 - y_0}{0.01} = e^{x_0}\cosh(2y_0 + x_0) \quad \therefore\ y_1 = 0.01\,e^{x_0}\cosh(2y_0 + x_0) + y_0$	M1 A1	
		$x_0 = 1,\ x_1 = 1.01;\ y_0 = 1 \quad \therefore\ y_1 = 1.2736... = 1.27$ (3sf)	A1	
	(b)	$\frac{y_1 - y_{-1}}{0.02} = e^{x_0}\cosh(2y_0 + x_0) \quad \therefore\ y_{-1} = y_1 - 0.02\,e^{x_0}\cosh(2y_0 + x_0)$	M1 A1	
		$x_{-1} = 0.99,\ x_0 = 1,\ x_1 = 1.01;\ y_0 = 1,\ y_1 = 1.2736...,\ y_{-1} = ?$		
		$\therefore\ y_{-1} = 0.7263... = 0.726$ (3sf)	A1	**(6)**

2.	*(a)*	$\overrightarrow{AB} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k},\ \ \overrightarrow{AC} = (a-2)\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$	B1	
		$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -1 \\ a-2 & -6 & 2 \end{vmatrix}$		
		$= \mathbf{i}(6-6) - \mathbf{j}(-8 + a - 2) + \mathbf{k}[24 - 3(a-2)] = (10 - a)\mathbf{j} + 3(10 - a)\mathbf{k}$	M1 A2	
	(b)	area $= \frac{1}{2}\,\lvert \overrightarrow{AB} \times \overrightarrow{AC} \rvert = 4\sqrt{10}$	M1	
		$\therefore\ \lvert 10 - a \rvert \times \sqrt{(1 + 9)} = 8\sqrt{10}$	M1	
		$\lvert 10 - a \rvert = 8$ so $a = 2$ or 18	A1	**(7)**

3.	*(a)*	$z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$	M1	
		$= \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta = 2\cos n\theta$	A1	
	(b)	dividing by z^2 gives $5z^2 - 11z + 16 - \frac{11}{z} + \frac{5}{z^2} = 0$	M1	
		$\therefore\ 5(2\cos 2\theta) - 11(2\cos\theta) + 16 = 0$	M1	
		$5\cos 2\theta - 11\cos\theta + 8 = 0$	A1	
		$5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$	M1	
		$10\cos^2\theta - 11\cos\theta + 3 = 0$		
		$(5\cos\theta - 3)(2\cos\theta - 1) = 0$	M1	
		$\therefore\ \cos\theta = \frac{3}{5}$ or $\frac{1}{2}$	A1	
		if $\cos\theta = \frac{3}{5}$, $\sin\theta = \pm\frac{4}{5}$; if $\cos\theta = \frac{1}{2}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$	M1	
		$\therefore\ z = \frac{3}{5} \pm \frac{4}{5}\mathrm{i}$ or $\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$	A1	**(10)**

4. *(a)* assume true for $n = k$ $\therefore$ $\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore$ $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+k & 1+k+\frac{1}{2}k(k+1) \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$ M1 A1

$1 + k + \frac{1}{2}k(k + 1) = \frac{1}{2}(k + 1)(2 + k) = \frac{1}{2}(k + 1)[(k + 1) + 1]$ M1

$\therefore$ $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k+1)[(k+1)+1] \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$ A1

$\therefore$ true for $n = k + 1$ if true for $n = k$

if $n = 1$ $\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}\times1\times2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore$ true for $n = 1$ B1

$\therefore$ by induction true for $n \in \mathbb{Z}^+$ A1

(b) $\det \mathbf{A}^n = 1(1 - 0) - n(0 - 0) + \frac{1}{2}n(n + 1)(0 - 0) = 1$ M1 A1

matrix of cofactors: $\begin{pmatrix} 1 & 0 & 0 \\ -n & 1 & 0 \\ \frac{1}{2}n(n-1) & -n & 1 \end{pmatrix}$ $[n^2 - \frac{1}{2}n(n + 1) = \frac{1}{2}n(n - 1)]$ M1 A1

$\therefore$ $(\mathbf{A}^n)^{-1} = \begin{pmatrix} 1 & -n & \frac{1}{2}n(n-1) \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix}$ A1 **(11)**

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5. *(a)* let $y = \arccos x$ $\therefore$ $\cos y = x$

$-\sin y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ M1

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-1}{\sin y} = \dfrac{-1}{\sqrt{1-x^2}}$ $\therefore$ $\mathrm{f}'(x) = \dfrac{-1}{(1-x^2)^{\frac{1}{2}}}$ M1 A1

(b) $\mathrm{f}''(x) = \frac{1}{2}(-2x)(1 - x^2)^{-\frac{3}{2}} = \dfrac{-x}{(1-x^2)^{\frac{3}{2}}}$ M1 A1

$\therefore$ $(1 - x^2)\mathrm{f}''(x) - x\mathrm{f}'(x) = (1 - x^2) \times \dfrac{-x}{(1-x^2)^{\frac{3}{2}}} + \dfrac{x}{(1-x^2)^{\frac{1}{2}}}$

$= \dfrac{-x}{(1-x^2)^{\frac{1}{2}}} + \dfrac{x}{(1-x^2)^{\frac{1}{2}}} = 0$ A1

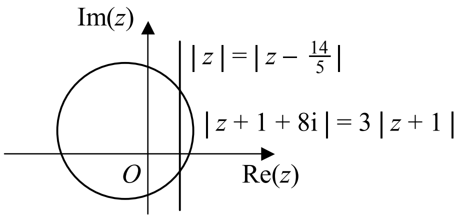
(c) $(1 - x^2)\mathrm{f}'''(x) - 2x\mathrm{f}''(x) - x\mathrm{f}''(x) - \mathrm{f}'(x) = 0$ M1 A1

$\mathrm{f}(0) = \frac{\pi}{2}$, $\mathrm{f}'(0) = -1$, $\mathrm{f}''(0) = 0$, $\mathrm{f}'''(0) = -1$ A1

$\therefore$ $\mathrm{f}(x) = \frac{\pi}{2} - 1x + 0 - 1(\frac{1}{3!})x^3 + \ldots = \frac{\pi}{2} - x - \frac{1}{6}x^3 + \ldots$ M1 A1 **(11)**

6.	*(a)*	$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$	M1	
		$(2-\lambda)[-\lambda(2-\lambda)-1] + 1[-(2-\lambda)-1] + 1(-1+\lambda) = 0$	A1	
		$(2-\lambda)(\lambda^2 - 2\lambda - 1) - (2-\lambda) - 1 - 1 + \lambda = 0$	M1	
		$(2-\lambda)(\lambda^2 - 2\lambda - 1) - (2-\lambda) - (2-\lambda) = 0$		
		$(2-\lambda)(\lambda^2 - 2\lambda - 1 - 2) = (2-\lambda)(\lambda^2 - 2\lambda - 3) = 0$	A1	
		$(2-\lambda)(\lambda-3)(\lambda+1) = 0$	M1	
		$\therefore\ \lambda = 2$ is an eigenvalue, also $\lambda = -1$ or 3	A2	
	(b)	$\lambda = 2,\ \begin{pmatrix} 0 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	M1	
		$-y + z = 0\ \therefore\ y = z;\quad x + y = 0\ \therefore\ x = -y$	M1 A1	
		$\therefore$ eigenvector $k\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	A1	
	(c)	$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	M1 A2	**(14)**

7.	*(a)*	$\lvert x + iy + 1 + 8i \rvert = 3\lvert x + iy + 1 \rvert$		
		$(x+1)^2 + (y+8)^2 = 9(x+1)^2 + 9y^2$	M1 A1	
		$y^2 + 16y + 64 = 8(x+1)^2 + 9y^2$	M1	
		$8(x+1)^2 + 8y^2 - 16y - 64 = 0$		
		$(x+1)^2 + y^2 - 2y - 8 = 0$		
		$(x+1)^2 + (y-1)^2 - 1 - 8 = 0$	M1	
		$(x+1)^2 + (y-1)^2 = 9$	A1	
		$\therefore$ circle, centre $-1 + i$, radius 3	A2	
	(b)	$\lvert z \rvert = \lvert z - \frac{14}{5} \rvert$ is perp. bisector of 0 and $\frac{14}{5}$ i.e. $\text{Re}(z) = \frac{7}{5}$	B1	



B1 B2

	(c)	intersect when $x = \frac{7}{5}$	M1	
		$\therefore\ (\frac{7}{5} + 1)^2 + (y-1)^2 = 9$	A1	
		$(y-1)^2 = 9 - \frac{144}{25} = \frac{81}{25}$	M1	
		$\therefore\ y = 1 \pm \frac{9}{5}$	A1	
		intersect at $\frac{7}{5} - \frac{4}{5}i$ and $\frac{7}{5} + \frac{14}{5}i$	A1	**(16)**

Total **(75)**

Performance Record – FP3 Paper C

Question no.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total
Topic(s)	step-by-step soln. of diff. eqn.	vectors	complex nos., De Moivre's	proof by induction, matrices	Maclaurin series	matrices, eigenvals., diag'lise	complex loci	
Marks	6	7	10	11	11	14	16	75
Student								