

June 2006
6675 Further Pure Mathematics FP2
Mark Scheme

Question number	Scheme	Marks
1.	$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$ $3e^{2x} - 22e^x + 7 = 0 \quad \text{M: Simplify to form quadratic in } e^x.$ $(3e^x - 1)(e^x - 7) = 0 \quad e^x = \frac{1}{3}, \quad e^x = 7 \quad \text{M: Solve 3 term quadratic.}$ $x = \ln \frac{1}{3} \text{ (or } -\ln 3) \quad x = \ln 7$	<p>B1</p> <p>M1 A1</p> <p>M1 A1</p> <p>A1 (6)</p> <p>6</p>
2.	<p>(a) Using $b^2 = a^2(1 - e^2)$ or equiv. to find e or ae: ($a = 2$ and $b = 1$) $e = \frac{\sqrt{3}}{2}$</p> <p>Using $y^2 = 4(ae)x$ $y^2 = 4\sqrt{3}x$ (M requires <u>values</u> for a and e)</p> <p>(b) $x = -\sqrt{3}$</p>	<p>M1 A1</p> <p>M1 A1 (4)</p> <p>B1ft (1)</p> <p>5</p>
3.	$\rho = \frac{ds}{d\psi} \quad s = \int e^{\sin\psi} \cos\psi d\psi = e^{\sin\psi} \quad (+k)$ <p>(M mark may be scored by a full substitution method)</p> $s = 0 \text{ at } \psi = 0: \quad k = -1 \quad s = e^{\sin\psi} - 1$ <p>(M mark requires a k, or use of limits)</p>	<p>M1 A1</p> <p>M1 A1cso (4)</p> <p>4</p>

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4.	$\frac{dy}{dx} = \frac{2x}{1+(x^2)^2} \qquad \text{M: } \frac{dy}{dx} = \frac{*}{1+(x^2)^2} \text{ or } \frac{dy}{dx} = \frac{2x}{1+x^2}$ $\frac{d^2y}{dx^2} = \frac{2(1+x^4) - 2x \cdot 4x^3}{(1+x^4)^2} \left(= \frac{2-6x^4}{(1+x^4)^2} \right)$ $\rho = \frac{\left(1 + \left(\frac{dy}{dx} \right)^2 \right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{\frac{3}{2}}}{-1}$ $= -2\sqrt{2} \quad (\text{or } 2\sqrt{2}) \quad (\text{or exact equivalent, e.g. } \sqrt{8}, 2^{\frac{3}{2}})$	<p>M1 A1</p> <p>M1</p> <p>M1 A1cso</p> <p>A1cso (6) 6</p>
	<p>2nd M: Quotient or product rule attempt. (Chain rule, if used, must be ‘good’.)</p> <p>3rd M: Attempt ρ with derivative values...</p> <p>A: Correct derivative values (1 and -1) seen or implied by working.</p> <p><u>Alternative:</u> (involving implicit differentiation).</p> <p>$\sec^2 y \frac{dy}{dx} = 2x$ [M1 A1] (allow one ‘slip’ for M1)</p> <p>$\sec^2 y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2 \sec y (\sec y \tan y) \frac{dy}{dx} = 2$ [M1]</p> <p>(or alternative e.g. $\frac{dy}{dx} = 2x \cos^2 y$, so $\frac{d^2y}{dx^2} = 2 \cos^2 y + 2x \cdot 2 \cos y (-\sin y) \frac{dy}{dx}$)</p> <p>Then marks as in main scheme (n.b. $\tan y = 1$, $\sec^2 y = 2$).</p>	

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5.	<p>(a) $\frac{dy}{dx} = 4\operatorname{sech}^2 4x - 1$</p> <p>Put $\frac{dy}{dx} = 0$ $(\cosh^2 4x = 4 \quad \cosh 4x = 2)$</p> <p>$4x = \ln(2 \pm \sqrt{3})$ or $8x = \ln(7 \pm 4\sqrt{3})$ or $e^{4x} = 2 \pm \sqrt{3}$ or $e^{4x} = 7 \pm 4\sqrt{3}$ (\pm or $+$)</p> <p>$x = \frac{1}{4}\ln(2 + \sqrt{3})$ or $x = \frac{1}{8}\ln(7 + 4\sqrt{3})$ (or equiv.)</p> <p>(b) $y = -\frac{1}{4}\ln(2 + \sqrt{3}) + \tanh(\dots)$ (Substitute for x)</p> <p>$\operatorname{sech} 4x = \frac{1}{2} = \sqrt{1 - \tanh^2 4x}, \quad \tanh 4x = \frac{\sqrt{3}}{2}$</p> <p>$y = \frac{\sqrt{3}}{2} - \frac{1}{4}\ln(2 + \sqrt{3}) = \frac{1}{4}\{2\sqrt{3} - \ln(2 + \sqrt{3})\}$ (*)</p>	<p>B1</p> <p>M1</p> <p>A1</p> <p>A1 (4)</p> <p>M1</p> <p>M1</p> <p>A1 (3)</p> <p>7</p>
	<p>(a) 'Second solution', if seen, must be rejected to score the final mark.</p> <p>(b) 2nd M requires an expression in terms of $\sqrt{3}$ without hyperbolics, exponentials and logarithms.</p>	

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6.	<p>(a) $\frac{dx}{dt} = 1 - \frac{1}{t}$ $\frac{dy}{dt} = 2t^{-\frac{1}{2}}$</p> $\sqrt{\left(1 - \frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}, \quad = \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} = 1 + \frac{1}{t} \text{ or } \frac{t+1}{t}$ $\text{Length} = \int_1^4 \left(1 + \frac{1}{t}\right) dt = [t + \ln t]_1^4 = (4 + \ln 4) - 1 = 3 + \ln 4 \quad (*)$ <p>(b) Surface area = $2\pi \int_1^4 4\sqrt{t} \sqrt{\left(1 - \frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2} dt \quad \left(= 8\pi \int_1^4 \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt\right)$</p> $= (8\pi) \left[\frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}} \right]_1^4 = (8\pi) \left\{ \left(\frac{16}{3} + 4 \right) - \left(\frac{2}{3} + 2 \right) \right\} = \frac{160\pi}{3} \quad \left(53\frac{1}{3}\pi \right)$	<p>B1 B1</p> <p>M1, A1</p> <p>M1 M1 A1 (7)</p> <p>M1</p> <p>M1 M1 A1 (4)</p> <p>11</p>
	<p>Note dependence of M's.</p> <p>(a) 2nd M: Complete integration attempt. 3rd M: Subs. correct limits and subtract.</p> <p>(b) 2nd M: Complete integration attempt. 3rd M: Subs. correct limits and subtract.</p> <p>In (b), missing 2 or π or 2π throughout could score M0 M1 M1 A0.</p> <p>If 2π is there initially, then lost, could score M1 M1 M1 A0.</p>	

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7.	$\int x^2 \operatorname{arsinh} x dx = \frac{x^3}{3} \operatorname{arsinh} x - \int \frac{x^3}{3\sqrt{x^2+1}} dx$ $\left[\frac{x^3}{3} \operatorname{arsinh} x \right]_0^3 = 9 \operatorname{arsinh} 3 \quad (\text{or } 9 \ln(3 + \sqrt{10}))$ <p>Let $u = x^2 + 1 \quad \frac{du}{dx} = 2x \quad \left[u^2 = x^2 + 1 \quad 2u \frac{du}{dx} = 2x \right]$</p> $\frac{1}{3} \int \frac{x^3}{u^2 \cdot 2x} du = \frac{1}{6} \int \frac{u-1}{u^2} du = \frac{1}{6} \int \left(u^{-1} - u^{-2} \right) du \quad \left[\frac{1}{3} \int (u^2 - 1) du \right]$ $= \frac{1}{6} \left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} \right] \quad \left[= \frac{1}{3} \left[\frac{u^3}{3} - u \right] \right]$ <p>When $x = 0, u = 1$ and when $x = 3, u = 10 \quad [\dots u = \sqrt{10}]$</p> $\frac{1}{6} \left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} \right]_1^{10} = \frac{1}{6} \left\{ \left(\frac{20\sqrt{10}}{3} - 2\sqrt{10} \right) - \left(\frac{2}{3} - 2 \right) \right\}$ $\text{Area} = 9 \operatorname{arsinh} 3 - \frac{1}{6} \left(\frac{14\sqrt{10}}{3} + \frac{4}{3} \right) = 9 \ln(3 + \sqrt{10}) - \frac{1}{9} (7\sqrt{10} + 2) \quad (*)$	<p>M1 A1 A1</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>A1cso (10)</p> <p>(10)</p>
	<p><u>Dependent M marks:</u></p> <p>M: Choose an appropriate substitution & find $\frac{du}{dx}$ <u>or</u> 'Set up' integration by parts.</p> <p>M: Get <u>all</u> in terms of 'u' <u>or</u> Use integration by parts.</p> <p>M: Sound integration.</p> <p>M: Substitute both limits (for the correct variable) and subtract.</p>	

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7.	<p><u>Alternative solution:</u></p> <p>Let $x = \sinh \theta \quad \frac{dx}{d\theta} = \cosh \theta$</p> $\int x^2 \operatorname{ar sinh} x dx = \int \theta \sinh^2 \theta \cosh \theta d\theta$ $= \left[\frac{\theta \sinh^3 \theta}{3} \right] - \int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) d\theta$ $\left[\frac{\theta \sinh^3 \theta}{3} \right]_0^{\operatorname{ar sinh} 3} = 9 \operatorname{ar sinh} 3$ $\int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) d\theta = \frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]$ $\frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]_0^{\operatorname{ar sinh} 3} = \frac{1}{3} \left\{ \left(\frac{10\sqrt{10}}{3} - \sqrt{10} \right) - \left(\frac{1}{3} - 1 \right) \right\}$ $\text{Area} = 9 \operatorname{ar sinh} 3 - \frac{1}{3} \left(\frac{7\sqrt{10}}{3} + \frac{2}{3} \right) = 9 \ln(3 + \sqrt{10}) - \frac{1}{9} (7\sqrt{10} + 2) \quad (*)$	<p>M1</p> <p>M1</p> <p>M1 A1 A1</p> <p>B1</p> <p>M1</p> <p>M1 A1</p> <p>A1cso (10)</p> <p>(10)</p>

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7.	<p>A few alternatives for: $\int \frac{x^3}{\sqrt{x^2+1}} dx$.</p> <p>(i) Let $u = x^2$ $\frac{du}{dx} = 2x$</p> $\int \frac{u^{\frac{3}{2}}}{\sqrt{1+u}} \cdot \frac{1}{2u^{\frac{1}{2}}} du = \frac{1}{2} \int \frac{u}{\sqrt{1+u}} du$ <p>No marks yet... needs another substitution, or parts, or perhaps...</p> $\frac{u}{\sqrt{1+u}} = \sqrt{1+u} - \frac{1}{\sqrt{1+u}}$ $\frac{1}{2} \int \sqrt{1+u} du - \frac{1}{2} \int \frac{1}{\sqrt{1+u}} du$ $\frac{1}{3} (1+u)^{\frac{3}{2}} - (1+u)^{\frac{1}{2}}$ <p>Limits (0 to 9)</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>
	<p>(ii) Let $x = \sinh \theta$ $\frac{dx}{d\theta} = \cosh \theta$</p> $\int \frac{\sinh^3 \theta}{\cosh \theta} \cdot \cosh \theta d\theta = \int \sinh \theta (\cosh^2 \theta - 1) d\theta$ <p>Then, as in the alternative solution,</p> $\int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) d\theta = \frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]$ <p>Limits (0 to arsinh3)</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>

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7.	(iii) Let $u = \tan \theta$ $\frac{du}{d\theta} = \sec^2 \theta$ $\int \frac{\tan^3 \theta}{\sec \theta} \cdot \sec^2 \theta d\theta = \int \tan \theta \sec \theta (\sec^2 \theta - 1) d\theta$ $= \int \sec^2 \theta (\sec \theta \tan \theta) d\theta - \int (\sec \theta \tan \theta) d\theta = \frac{\sec^3 \theta}{3} - \sec \theta$ Limits ($\sec \theta = 1$ to $\sec \theta = \sqrt{10}$)	M1 M1 M1 M1
	(iv) (By parts... must be the 'right way round', not integrating x^2) $u = x^2, \frac{du}{dx} = 2x$ $\frac{dv}{dx} = \frac{x}{\sqrt{1+x^2}}, v = \sqrt{1+x^2}$ $x^2 \sqrt{1+x^2} - \int 2x \sqrt{1+x^2} dx$ $x^2 \sqrt{1+x^2} - \frac{2}{3} (x^2 + 1)^{\frac{3}{2}}$ Limits	M1 M1 M1 M1
	(v) (By parts) $u = x^3, \frac{du}{dx} = 3x^2$ $\frac{dv}{dx} = \frac{1}{\sqrt{1+x^2}}, v = \operatorname{arsinh} x$	No progress M0
	(vi) $\frac{x^3}{\sqrt{1+x^2}} = \frac{x(x^2+1) - x}{\sqrt{1+x^2}} = \frac{x(x^2+1)}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}$ $\int x \sqrt{1+x^2} dx - \int \frac{x}{\sqrt{1+x^2}} dx$ $= \frac{1}{3} (1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}$ Limits	M1 M1 M1 M1

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8.	<p>(a) $\int x^n \cosh x dx = x^n \sinh x - \int nx^{n-1} \sinh x dx$</p> <p>$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1) \int x^{n-2} \cosh x dx$</p> <p>$I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2}$ (*)</p> <p>(b) $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2$</p> <p>$I_4 = x^4 \sinh x - 4x^3 \cosh x + 12(x^2 \sinh x - 2x \cosh x + 2I_0)$</p> <p>(This M may also be scored by finding I_2 by integration.)</p> <p>$I_0 = \int \cosh x dx = \sinh x + k$</p> <p>$I_4 = (x^4 + 12x^2 + 24)\sinh x, + (-4x^3 - 24x)\cosh x (+C)$</p> <p>(c) $[(x^4 + 12x^2 + 24)\sinh x + (-4x^3 - 24x)\cosh x]_0^1$</p> <p>$= 37\sinh 1 - 28\cosh 1$ M: $x = 1$ substituted throughout (at some stage)</p> <p>$= 37\left(\frac{e - e^{-1}}{2}\right) - 28\left(\frac{e + e^{-1}}{2}\right)$</p> <p>M: Use of exp. Definitions (can be in terms of x)</p> <p>$= \frac{1}{2}(9e - 65e^{-1})$</p>	<p>M1 A1</p> <p>M1</p> <p>A1 (4)</p> <p>M1</p> <p>M1</p> <p>B1</p> <p>A1, A1 (5)</p> <p>M1</p> <p>M1</p> <p>A1 (3)</p> <p>12</p>
	(b) Integration constant missing <u>throughout</u> loses the B mark.	

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9.	<p>(a) $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$ $b^2x^2 + a^2(mx+c)^2 = a^2b^2$</p> <p>$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$ (*)</p> <p>(b) $(2a^2mc)^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$</p> <p>$4a^4m^2c^2 = 4a^2(b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2)$</p> <p>$c^2 = b^2 + a^2m^2$ (*)</p> <p>(c) Find height and base of triangle (perhaps in terms of c).</p> <p>$OB = c \left(= \sqrt{b^2 + a^2m^2} \right)$ and $AO = \frac{c}{m} \left(= \frac{\sqrt{b^2 + a^2m^2}}{m} \right)$</p> <p>Area of triangle $OAB = \frac{c^2}{2m} = \frac{b^2 + a^2m^2}{2m}$ M: Find area and subs. for c.</p> <p>(d) $\Delta = \frac{b^2 + a^2m^2}{2m} = \frac{b^2}{2}m^{-1} + \frac{a^2}{2}m$</p> <p>$\frac{d\Delta}{dm} = -\frac{b^2}{2}m^{-2} + \frac{a^2}{2} = 0$ $\frac{b^2}{m^2} = a^2$ $m = \frac{b}{a}$</p> <p>$\Delta = \left(\frac{b^2}{2}\right)\left(\frac{a}{b}\right) + \left(\frac{a^2}{2}\right)\left(\frac{b}{a}\right) = ab$ (*)</p> <p>(e) Root of quadratic: $x = \frac{-a^2mc}{b^2 + a^2m^2}$ (Should be <u>correct</u> if quoted directly)</p> <p>Using $m = \frac{b}{a}$ and $c = \sqrt{b^2 + a^2m^2}$: $x = -\frac{a}{\sqrt{2}}$</p> <p>(The 2nd M is dependent on using the quadratic equation).</p>	<p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1</p> <p>M1 A1 (4)</p> <p>M1 A1</p> <p>A1 (3)</p> <p>M1</p> <p>M1 A1 (3)</p> <p>14</p>
	<p>(d) <u>Alternative</u>: $b^2 + a^2m^2 \geq 2bam$ (since $(b-am)^2 \geq 0$) [M1]</p> <p>$\frac{b^2 + a^2m^2}{2m} \geq ab$ [A1] ...Conclusion [A1]</p> <p>(e) <u>Alternative</u>: Begin with full eqn. $(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$.</p> <p>In the eqn., use conditions $m = \frac{b}{a}$ and $c = \sqrt{b^2 + a^2m^2} (= b\sqrt{2})$ [M1]</p> <p>Simplify and solve eqn., e.g. $2x^2 + 2a\sqrt{2}x + a^2 = 0$ $x = -\frac{a}{\sqrt{2}}$</p>	